

Dynamics in Non-Globally-Hyperbolic Static Spacetimes II: General Analysis of Prescriptions for Dynamics

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Abstract

It was previously shown by one of us that in any static, non-globally-hyperbolic, spacetime it is always possible to define a sensible dynamics for a Klein-Gordon scalar field. The prescription proposed for doing so involved viewing the spatial derivative part, A , of the wave operator as an operator on a certain L^2 Hilbert space \mathcal{H} and then defining a positive, self-adjoint operator on \mathcal{H} by taking the Friedrichs extension (or other positive extension) of A . However, this analysis left open the possibility that there could be other inequivalent prescriptions of a completely different nature that might also

yield satisfactory definitions of the dynamics of a scalar field. We show here that this is not the case. Specifically, we show that if the dynamics agrees locally with the dynamics defined by the wave equation, if it admits a suitable conserved energy, and if it satisfies certain other specified conditions, then it must correspond to the dynamics defined by choosing some positive, self-adjoint extension of A on \mathcal{H} . Thus, subject to our requirements, the previously given prescription is the only possible way of defining the dynamics of a scalar field in a static, non-globally-hyperbolic, spacetime. In a subsequent paper, this result will be applied to the analysis of scalar, electromagnetic, and gravitational perturbations of anti-de Sitter spacetime. By doing so, we will determine all possible choices of boundary conditions at infinity in anti-de Sitter spacetime that give rise to sensible dynamics.

1 Introduction

Let (M, g_{ab}) be a spacetime and let Φ be a real¹ scalar field in this spacetime that satisfies the Klein-Gordon equation

$$\nabla^a \nabla_a \Phi - m^2 \Phi = 0. \quad (1)$$

It is well known (see, e.g., [1]) that if (M, g_{ab}) is globally hyperbolic and if Σ is a Cauchy surface for (M, g_{ab}) , then for each $(\phi_0, \dot{\phi}_0) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$, there exists a unique $\Phi \in C^\infty(M)$ that satisfies eq. (1) and is such that $\Phi|_\Sigma = \phi_0$ and $t^a \nabla_a \Phi|_\Sigma = \dot{\phi}_0$, where t^a is a given (not necessarily unit) normal to Σ .

However, no similar result holds, in general, when (M, g_{ab}) fails to be globally hyperbolic. Given suitable smooth data for Φ at some “time” (i.e., on a spacelike slice, Σ), the partial differential equation (1) need not, in general, admit any solutions corresponding to this data and, even when a solution exists, it need not be unique. Given the infinite variety of singularities and causal pathologies that can occur in an arbitrary non-globally-hyperbolic spacetime, it is far from clear how to even attempt to give a unique prescription for defining dynamics. Nevertheless, it is of interest to analyze dynamics in non-globally-hyperbolic spacetimes and determine whether there exist prescriptions that give rise to acceptable, unique dynamics, since this might give

¹The analysis of the dynamics of a complex scalar field follows immediately from that of the real scalar field by treating the real and imaginary parts separately.

hints as to whether and how the spacetime singularities themselves might be “resolved” in quantum gravity. It is well known that classical general relativity predicts the occurrence of singularities. If these singularities are “resolved” by quantum gravity, then one might expect to have well defined, deterministic predictions in situations corresponding to the presence of classical singularities.

As a first step in the direction of analyzing dynamics in non-globally-hyperbolic spacetimes, attention was restricted in [2] to static, non-globally-hyperbolic spacetimes that possess a hypersurface Σ orthogonal to the static Killing field, t^a , such that the orbits of t^a are complete and each orbit intersects Σ once and only once. No causal pathologies occur in such spacetimes (since they are automatically stably causal), but there is still an infinite variety of possible singular behaviour. It was shown in [2] that in such spacetimes, a sensible dynamics always can be defined (for sufficiently nice initial data) in the following manner:

In a static spacetime, the Klein-Gordon equation (1) can be written in the form

$$\partial^2 \Phi / \partial t^2 = -A\Phi, \quad (2)$$

where t denotes the Killing parameter and

$$A = -VD^a(VD_a) + m^2V^2, \quad (3)$$

where $V = (-t^a t_a)^{1/2}$ and D_a is the derivative operator on Σ associated with the induced metric on Σ . View A as an operator (with domain $C_0^\infty(\Sigma)$) on the Hilbert space $\mathcal{H} = L^2(\Sigma, \mu)$, where the measure μ on Σ is chosen to be V^{-1} times the natural volume element on Σ associated with the induced metric. Then A is a positive, symmetric operator on \mathcal{H} . Consequently, it admits at least one positive self-adjoint extension—namely, the Friedrichs extension. Choose some positive, self-adjoint extension, A_E . Given $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$, for each $t \in \mathbf{R}$ we define

$$\phi_t = \cos(A_E^{1/2}t)\phi_0 + A_E^{-1/2}\sin(A_E^{1/2}t)\dot{\phi}_0. \quad (4)$$

Here $\cos(A_E^{1/2}t)$ and $A_E^{-1/2}\sin(A_E^{1/2}t)$ are bounded operators defined via the spectral theorem. Since, clearly, we have $(\phi_0, \dot{\phi}_0) \in \mathcal{H} \times \mathcal{H}$, it follows that for all t , ϕ_t is a well defined element of \mathcal{H} . It was shown in [2] that there exists a unique $\Phi \in C^\infty(M)$ such that for all t , $\Phi|_{\Sigma_t} = \phi_t$ and $t^a \nabla_a \Phi|_{\Sigma_t} = \dot{\phi}_t$,

where Σ_t denotes the time translate of Σ by t . Furthermore, Φ satisfies eq. (1) throughout M , and satisfies the additional properties enumerated in the next section. Therefore, eq. (4) provides a satisfactory prescription for defining the dynamics associated with eq. (1), at least for initial data in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$.

Note that different choices of self-adjoint extensions, A_E , give rise to inequivalent prescriptions for defining dynamical evolution. (In essence, they correspond to inequivalent choices of “boundary conditions at the singularity”.) Thus, if A has more than one positive self-adjoint extension, there will be more than one choice of dynamical evolution law as defined by the prescription eq. (4). On the other hand, if A has a unique self-adjoint extension—as occurs in some cases of interest [3, 4]—the general prescription eq. (4) yields unique evolution. However, the analysis of [2] leaves open the possibility that, in addition to the freedom of choosing different self-adjoint extensions of A , there could exist other acceptable prescriptions for defining dynamics that are not even of the form of eq. (4). If that were the case, one could not conclude that dynamical evolution is uniquely defined even in situations where A has a unique self-adjoint extension.

The main purpose of this paper is to give a general analysis of the possibilities for defining dynamics in static, non-globally-hyperbolic spacetimes, and show that—subject to certain requirements—the above prescription eq. (4) is the only possible one. In the next section, we shall specify the requirements that we impose upon the dynamics. In section 3, we shall then prove that these requirements imply that the dynamics is of the form eq. (4). Applications to defining dynamics in anti-de Sitter spacetime will be given in a subsequent paper.

2 Assumptions Concerning Dynamics

Let (M, g_{ab}) be a static (but non-globally-hyperbolic) spacetime, i.e., (M, g_{ab}) possesses an everywhere timelike hypersurface orthogonal Killing field, t^a , whose orbits are complete. We further assume that there exists a slice Σ orthogonal to t^a such that every orbit of t^a intersects Σ once and only once. Let Σ_t denote image of Σ under the static isometry by parameter t ; it follows that $\{\Sigma_t\}$ provides a foliation of M . We shall label each point $p \in M$ by (t, x) , where $x \in \Sigma$ denotes the intersection with Σ of the orbit of t^a through

p , and t denotes the parameter of p along this orbit starting from x . It follows that t is a global time function on M , and hence (M, g_{ab}) is stably causal.

We consider the Klein-Gordon equation (1), which, as noted above, can be written in the form eq. (2), with A being the differential operator given by eq. (3). We seek a rule for providing us with a solution to eq. (1) associated to any allowed initial data on Σ . We do not know in advance the widest possible class of initial data that should be “allowed”, but we demand that it at least include all data on Σ that is smooth and of compact support. Thus, our goal is the following: We seek a suitable prescription such that given any $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$, we obtain a unique $\Phi \in C^\infty(M)$ such that Φ depends linearly on $(\phi_0, \dot{\phi}_0)$, satisfies eq. (1) throughout M , and is such that $\Phi|_\Sigma = \phi_0$ and $t^a \nabla_a \Phi|_\Sigma = \dot{\phi}_0$. We now state the additional requirements that we impose upon the prescription so that it is “suitable”.

First, we require that the solution Φ must be compatible with causality in the following sense.

Assumption 1 (causality): Let K_0 denote the support of the initial data on Σ , i.e., let

$$K_0 = \text{supp}(\phi_0) \cup \text{supp}(\dot{\phi}_0). \quad (5)$$

Then we require that

$$\text{supp}(\Phi) \subset J^+(K_0) \cup J^-(K_0), \quad (6)$$

where J^\pm denotes the causal future/past.

Let Φ be the solution corresponding to the initial data $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$. Define

$$\phi_t \equiv \Phi|_{\Sigma_t}, \quad (7)$$

and

$$\dot{\phi}_t \equiv (\partial\Phi/\partial t)|_{\Sigma_t}. \quad (8)$$

The following lemma will be useful in many places in our analysis.

Lemma 2.1: Suppose that the dynamics satisfies assumption 1. Then there exists a $\delta > 0$ such that ϕ_t and $\dot{\phi}_t$ are each in $C_0^\infty(\Sigma_t)$ for all $|t| < \delta$.

Proof: Since Φ is smooth, clearly ϕ_t and $\dot{\phi}_t$ are smooth for all t , so we need only show that they have compact support for sufficiently small t . We show first that there exists a $\delta > 0$ such that $K_0 \subset \text{int}D(\Sigma_t)$ for all $|t| < \delta$, where

D denotes the domain of dependence. Namely, if not, we could find a sequence $\{x_n\} \in K_0$ such that $x_n \notin \text{int}D(\Sigma_{t_n})$ for all n , where the sequence $\{t_n\}$ converges to zero. Since K_0 is compact, there exists an accumulation point $x \in K_0$, and it follows that $x \notin \text{int}D(\Sigma_{t_n})$ for sufficiently large n . However, this is impossible, since clearly $x \in \text{int}D(\Sigma)$ and the time translation isometries are a continuous map from $\mathbf{R} \times M$ into M , so we must have $x \in \text{int}D(\Sigma_t)$ for sufficiently small t .

Now choose $|t| < \delta$; for definiteness we assume that $t \geq 0$. Then, by the above result, we have $J^+(K_0) \cap J^-(\Sigma_t) \subset \text{int}D(\Sigma_t)$, so we may restrict attention to the globally hyperbolic sub-spacetime $\text{int}D(\Sigma_t)$. But, since K_0 is compact, it follows from a slight generalisation of a standard theorem (see theorem 8.3.12 of [1]) that $J^+(K_0) \cap \Sigma_t$ is compact. It then follows immediately from assumption 1 above that the supports of ϕ_t and $\dot{\phi}_t$ are compact. \square

Our second set of requirements ensure that our prescription is compatible with the spacetime symmetries. Let $T_t : C^\infty(M) \rightarrow C^\infty(M)$ denote the natural action of the time translation isometries on smooth functions on M , i.e., for any smooth $F : M \rightarrow M$ we define

$$(T_t F)(s, x) = F(s - t, x). \quad (9)$$

Similarly, let $P : C^\infty(M) \rightarrow C^\infty(M)$ denote the natural action of the time reflection isometry on smooth functions on M , i.e., for any smooth $F : M \rightarrow M$ we define

$$(PF)(t, x) = F(-t, x). \quad (10)$$

We require our prescription to satisfy the following properties:

Assumption 2(i) (time translation invariance): Let Φ be the solution associated with the data $(\phi_0, \dot{\phi}_0)$ on Σ . Suppose that $K_t \equiv \text{supp}(\phi_t) \cup \text{supp}(\dot{\phi}_t)$ is compact—which, as shown above, it must be for all $|t| < \delta$ for some $\delta > 0$. We require that if the “time translated” data $(\phi_t, \dot{\phi}_t)$ is specified on Σ , then the corresponding solution must be the time translate, $T_{-t}\Phi$, of the original solution Φ . Furthermore, if the data $(\dot{\phi}_0, -A\phi_0)$ is specified on Σ —which formally corresponds to initial data for $\partial\Phi/\partial t$ —then we demand that the corresponding solution is $\partial\Phi/\partial t$.

Assumption 2(ii) (time reflection invariance): If the data $(\phi_0, -\dot{\phi}_0)$ is specified on Σ —which formally corresponds to initial data for the time reverse of Φ —then we require that the corresponding solution is $P\Phi$.

Our final set of requirements concern the existence of a suitable conserved “energy” for solutions. Consider, first, the case of a globally hyperbolic, static spacetime. In this case, if $(\phi_0, \dot{\phi}_0)$ is of compact support, then $(\phi_t, \dot{\phi}_t)$ will be of compact support for all t . For the unique solution, Φ , corresponding to this initial data, define

$$\begin{aligned} E(\Phi, \Phi) &= \int_{\Sigma} \dot{\phi}_0^2 V^{-1} d\Sigma + \int_{\Sigma} \phi_0 A \phi_0 V^{-1} d\Sigma \\ &= (\dot{\phi}_0, \dot{\phi}_0)_{L^2} + (\phi_0, A\phi_0)_{L^2}, \end{aligned} \quad (11)$$

where the L^2 inner product is defined using the volume element specified below eq. (3), and the integrals clearly converge since all functions appearing in the integrals are smooth and of compact support. It is easily seen that E is positive definite, and it follows immediately from the Klein-Gordon equation (2) that E is conserved in the sense that

$$E(T_t \Phi, T_t \Phi) = E(\Phi, \Phi). \quad (12)$$

We may therefore view E as a conserved inner product on the vector space, \mathcal{W} of solutions with initial data in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$. More generally, if Φ is a solution with initial data in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ and Ξ is a solution with initial data that is merely in $C^\infty(\Sigma) \times C^\infty(\Sigma)$, then

$$E(\Xi, \Phi) = (\dot{\xi}_0, \dot{\phi}_0)_{L^2} + (\xi_0, A\phi_0)_{L^2} \quad (13)$$

also is well defined and is conserved.

We shall require our prescription for defining dynamics in the non-globally-hyperbolic case to admit a conserved energy which reduces to eq. (13) in appropriate cases. In the non-globally-hyperbolic case, a solution with data $(\phi_0, \dot{\phi}_0)$ in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ will not, in general, be such that ϕ_t and $\dot{\phi}_t$ will be of compact support for all t (although, as shown above, ϕ_t and $\dot{\phi}_t$ will be of compact support for all $|t| < \delta$ for some $\delta > 0$). Consequently, if we define \mathcal{W} to be the vector space of solutions (as given by our prescription) with initial data $(\phi_0, \dot{\phi}_0)$ in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$, then T_t will not map \mathcal{W} into itself. It is therefore useful to enlarge the solution space \mathcal{W} as follows.

Let \mathcal{V} denote the vector space of solutions to the Klein-Gordon equation that can be expressed as finite linear combinations of solutions of the form $T_t \Phi$ for $\Phi \in \mathcal{W}$ and $t \in \mathbf{R}$, i.e.,

$$\mathcal{V} \equiv \{\Psi | \Psi = T_{t_1} \Phi_1 + \dots + T_{t_n} \Phi_n, \Phi_i \in \mathcal{W}\}. \quad (14)$$

(Note that in the globally hyperbolic case, we have $\mathcal{V} = \mathcal{W}$, but in the non-globally-hyperbolic case, in general, \mathcal{W} will be a proper subset of \mathcal{V} .) Define the initial data spaces, X and Y , for \mathcal{V} by

$$X \equiv \{\psi_0 = \Psi|_{\Sigma} | \Psi \in \mathcal{V}\}, \quad (15)$$

$$Y \equiv \{\dot{\psi}_0 = (\partial\Psi/\partial t)|_{\Sigma} | \Psi \in \mathcal{V}\}. \quad (16)$$

Clearly, we have $C_0^\infty(\Sigma) \subset X$ and $C_0^\infty(\Sigma) \subset Y$. Several additional properties of the spaces X and Y follow immediately from our previous assumptions. First, since the last condition of assumption 2(i) implies that if $\Psi \in \mathcal{V}$ then $\partial\Psi/\partial t \in \mathcal{V}$, it follows immediately that $Y \subset X$. Furthermore, assumption 2(ii) implies that if $\Psi \in \mathcal{V}$ then $P\Psi \in \mathcal{V}$, and consequently, $(\Psi \pm P\Psi) \in \mathcal{V}$. It follows that if $(\psi_0, \dot{\psi}_0)$ is initial data for a solution Ψ in \mathcal{V} , then $(\psi_0, 0)$ and $(0, \dot{\psi}_0)$ are also initial data for a solution \mathcal{V} . Consequently, the vector space of initial data for solutions in \mathcal{V} is isomorphic to $X \oplus Y$.

We require that our prescription for dynamics be such that there exist a symmetric, positive definite bilinear map (i.e., an inner product) $E : \mathcal{V} \times \mathcal{V} \rightarrow \mathbf{R}$ —which we may equivalently view as a bilinear map from $(X \oplus Y) \times (X \oplus Y)$ into \mathbf{R} —such that the following properties hold:

Assumption 3(i) (time translation invariance of E): E is time translation invariant in the sense that for all $\Psi_1, \Psi_2 \in \mathcal{V}$ and all t we have

$$E(T_t\Psi_1, T_t\Psi_2) = E(\Psi_1, \Psi_2). \quad (17)$$

Assumption 3(ii) (time reflection invariance of E): E is time reflection invariant in the sense that for all $\Psi_1, \Psi_2 \in \mathcal{V}$ we have

$$E(P\Psi_1, P\Psi_2) = E(\Psi_1, \Psi_2). \quad (18)$$

Assumption 3(iii) (agreement with formula in the globally hyperbolic case): If $\Phi \in \mathcal{W}$ and $\Psi \in \mathcal{V}$ (but Ψ need not be in \mathcal{W}) then E is given the same formula as in the globally hyperbolic case (see eq. (13) above), i.e.,

$$E(\Psi, \Phi) = (\dot{\psi}_0, \dot{\phi}_0)_{L^2} + (\psi_0, A\phi_0)_{L^2}. \quad (19)$$

Assumption 3(iv) (compatibility of convergence with respect to E with more elementary notions of convergence of initial data): Suppose that $\{\Psi_n\}$

is a sequence in \mathcal{V} that is a Cauchy with respect to the norm defined by E . Suppose, further, that there exists a $\Psi \in \mathcal{V}$ such that $(\psi_n)_0$ and all of its spatial derivatives converge uniformly on compact subsets of Σ to ψ_0 and its corresponding spatial derivatives and, similarly, that $(\dot{\psi}_n)_0$ and all of its spatial derivatives converge uniformly on compact subsets of Σ to $\dot{\psi}_0$ and its corresponding spatial derivatives. Then we require that $\{\Psi_n\}$ converge to Ψ in the norm defined by E , i.e., we require that

$$\lim_{n \rightarrow \infty} E(\Psi_n - \Psi, \Psi_n - \Psi) = 0. \quad (20)$$

3 General Analysis of Dynamics

In this section, we shall prove that any prescription for defining dynamics that satisfies the assumptions of the previous section must, in fact, be equivalent to the prescription of [2], obtained by choosing a self-adjoint extension of the operator A on $L^2(\Sigma, \mu)$.

It will be convenient to view the energy, E , introduced in the previous section, as a bilinear map $E : (X \oplus Y) \times (X \oplus Y) \rightarrow \mathbf{R}$ on initial data space. We may break E up into a sum of four maps of the form $Q : X \times X \rightarrow \mathbf{R}$, $S : Y \times Y \rightarrow \mathbf{R}$, $R : X \times Y \rightarrow \mathbf{R}$, and $T : Y \times X \rightarrow \mathbf{R}$, i.e., we may write

$$E([\psi_0, \dot{\psi}_0], [\chi_0, \dot{\chi}_0]) = Q(\psi_0, \chi_0) + S(\dot{\psi}_0, \dot{\chi}_0) + R(\dot{\psi}_0, \chi_0) + T(\psi_0, \dot{\chi}_0). \quad (21)$$

By assumption 3(ii), E is invariant under the time reflection map P . Since the action of P on initial data is to take $(\psi_0, \dot{\psi}_0)$ into $(\psi_0, -\dot{\psi}_0)$, it follows immediately that the time reflection invariance of E implies that $R = T = 0$. The positive definiteness of E on \mathcal{V} implies that Q defines an inner product on X and that S defines an inner product on Y . In fact, we now show that the inner product, S , on Y is just the L^2 inner product considered in the analysis of [2].

Lemma 3.1: Let Y and $S : Y \times Y \rightarrow \mathbf{R}$ be defined as above (see eqs. (16) and (21)). Then, under the assumptions stated in the previous section, we have $Y \subset L^2(\Sigma, \mu)$, where the measure, μ is that arising from V^{-1} times the natural volume element on Σ . Furthermore, for all $\xi, \eta \in Y$, we have

$$S(\xi, \eta) = (\xi, \eta)_{L^2}. \quad (22)$$

Proof: Let $\xi \in Y$ and consider the solution $\Psi \in \mathcal{V}$ with initial data ($\psi_0 = 0, \dot{\psi}_0 = \xi$). Let $\{O_n\}$ be a nested family of open subsets of Σ with compact closure such that $\cup_n O_n = \Sigma$. (Such a family can be constructed by putting a complete Riemannian metric on Σ , choosing a point $p \in \Sigma$, and taking O_n to be the open ball of radius n in this metric about point p .) Let $\{f_n\}$ be a sequence of functions in $C_0^\infty(\Sigma)$ with $0 \leq f_n \leq 1$ everywhere, and $f_n(x) = 1$ if $x \in O_{n-1}$ but $f_n(x) = 0$ if $x \notin O_{n+1}$. Let $\xi_n = f_n \xi$. Let $\Psi_n \in \mathcal{W}$ be the solution associated with the initial data $(0, \xi_n)$. Then we have

$$\begin{aligned}
S(\xi, \xi) &= E(\Psi, \Psi) \\
&= E(\Psi - \Psi_n, \Psi - \Psi_n) + E(\Psi - \Psi_n, \Psi_n) \\
&\quad + E(\Psi_n, \Psi - \Psi_n) + E(\Psi_n, \Psi_n) \\
&= E(\Psi - \Psi_n, \Psi - \Psi_n) + 2 \int_{\Sigma} f_n(1 - f_n) \xi^2 V^{-1} d\Sigma \\
&\quad + \int_{\Sigma} (f_n \xi)^2 V^{-1} d\Sigma \\
&\geq \int_{\Sigma} (f_n \xi)^2 V^{-1} d\Sigma,
\end{aligned} \tag{23}$$

where we have used assumption 3(iii) to obtain an explicit form for E in the case where one of the arguments, Ψ_n , lies in \mathcal{W} . However, the last inequality in eq. (23) implies that $\xi \in L^2(\Sigma, \mu)$, since, if not, given any $C > 0$ we could find an n such that $\|f_n \xi\|_{L^2}^2 > C$. This shows that $Y \subset L^2$. But, since $\xi \in L^2$, it follows that $\{\xi_n = f_n \xi\}$ is a Cauchy sequence in L^2 , i.e., $\{\Psi_n\}$ is a Cauchy sequence in the energy norm. Since $\{\xi_n\}$ and all spatial derivatives of $\{\xi_n\}$ clearly converge uniformly on all compact subsets of Σ to ξ and its corresponding spatial derivatives, it follows from assumption 3(iv) that $\{\Psi_n\}$ converges to Ψ in the energy norm. But this means that for all $\xi \in Y$ we have

$$S(\xi, \xi) = \lim_{n \rightarrow \infty} \int_{\Sigma} (f_n \xi)^2 V^{-1} d\Sigma = \|\xi\|_{L^2}^2, \tag{24}$$

which implies that the inner product on Y defined by S is just the L^2 inner product, as we desired to show. \square

Now complete \mathcal{V} in the inner product E to obtain the real Hilbert space \mathcal{H}_E . Similarly, we complete X and Y in the inner products Q and S , respectively, to obtain real Hilbert spaces \mathcal{H}_X and $\mathcal{H}_Y = L^2(\Sigma, \mu)$, respectively. Clearly, we have

$$\mathcal{H}_E \cong \mathcal{H}_X \oplus L^2(\Sigma, \mu). \tag{25}$$

Now, by assumption 3(i), for each t , $T_t : \mathcal{V} \rightarrow \mathcal{V}$ is a norm preserving—and, hence, bounded—linear map in the norm E , so it can be uniquely extended to a bounded linear map $T_t : \mathcal{H}_E \rightarrow \mathcal{H}_E$. By continuity, the extended T_t also is norm preserving. Furthermore, the range of T_t clearly includes \mathcal{V} , so the range of T_t is dense in \mathcal{H}_E . Consequently, $T_t : \mathcal{H}_E \rightarrow \mathcal{H}_E$ is unitary for each t . Furthermore, on \mathcal{V} we have $T_t \circ T_s = T_{t+s}$ and, by continuity, this relation must hold on \mathcal{H}_E . Thus, T_t is a one-parameter unitary group on \mathcal{H}_E . Furthermore, we have the following proposition:

Proposition 3.1: For all $\Psi \in \mathcal{V}$, $T_t\Psi$ is strongly differentiable in \mathcal{H}_E , and its derivative is $\partial\Psi/\partial t$.

Proof: Our task is to show that for all $\Psi \in \mathcal{V}$,

$$\lim_{t \rightarrow 0} \left\| \frac{T_t\Psi - \Psi}{t} - \frac{\partial\Psi}{\partial t} \right\|_E = 0. \quad (26)$$

Since \mathcal{V} consists of finite sums of time translates of elements of \mathcal{W} (see eq. (14) above), it suffices to prove eq. (26) for solutions of the form $T_s\Phi$ for $\Phi \in \mathcal{W}$. However, since T_t is a unitary group, it follows immediately that it suffices to prove eq. (26) for $\Phi \in \mathcal{W}$. In that case, we also have $\partial\Phi/\partial t \in \mathcal{W}$ and, by Lemma 2.1, there exists a $\delta > 0$ such that $T_t\Phi \in \mathcal{W}$ for all $|t| < \delta$. Consequently, by assumption 3(iii), the energy norm appearing in eq. (26) is given explicitly by eq. (19). The result then follows immediately from standard results for solutions with smooth data of compact support to the partial differential eq. (1) on the globally hyperbolic spacetime $\text{int}D(\Sigma)$. \square

Corollary: T_t defines a strongly continuous one-parameter group on \mathcal{H}_E .

Proof: We have already established that T_t is a one-parameter group on \mathcal{H}_E , so we need only show that it is strongly continuous. However, we have just shown that $T_t\Psi$ is strongly differentiable—and, hence, strongly continuous—for all $\Psi \in \mathcal{V}$. Since \mathcal{V} is dense in \mathcal{H}_E , given $\Xi \in \mathcal{H}_E$ and given $\epsilon > 0$ we can find a $\Psi \in \mathcal{V}$ such that $\|\Xi - \Psi\|_E < \epsilon/3$. Writing

$$T_t\Xi - \Xi = T_t(\Xi - \Psi) - (\Xi - \Psi) + (T_t\Psi - \Psi), \quad (27)$$

we see immediately that

$$\|T_t\Xi - \Xi\|_E \leq 2\epsilon/3 + \|T_t\Psi - \Psi\|_E \quad (28)$$

so strong continuity of T_t on \mathcal{V} implies strong continuity of T_t on \mathcal{H}_E . \square

Since T_t is a strongly continuous one-parameter unitary group on \mathcal{H}_E , by Stone's theorem (adapted here to the real Hilbert space case), there exists a skew adjoint map $\mathcal{B} : \mathcal{H}_E \rightarrow \mathcal{H}_E$ such that $T_t = \exp(-t\mathcal{B})$. Furthermore, since $T_t\Psi$ is strongly differentiable for all $\Psi \in \mathcal{V}$ we have $\mathcal{V} \subset \text{Dom}(\mathcal{B})$. Since we have shown in Proposition 3.1 above that for all $\Psi \in \mathcal{V}$ the derivative of $T_t\Psi$ is just $\partial\Psi/\partial t$, we have for all $\Psi \in \mathcal{V}$

$$\frac{\partial\Psi}{\partial t} = -\mathcal{B}\Psi. \quad (29)$$

Using the isomorphism (25), we can view \mathcal{B} as a skew-adjoint operator on $\mathcal{H}_X \oplus L^2(\Sigma, \mu)$ and re-write eq. (29) as the pair of equations

$$\dot{\psi}_0 = -\mathcal{B}_{11}\psi_0 - \mathcal{B}_{12}\dot{\psi}_0, \quad (30)$$

$$\ddot{\psi}_0 = -\mathcal{B}_{21}\psi_0 - \mathcal{B}_{22}\dot{\psi}_0, \quad (31)$$

where

$$\ddot{\psi}_0 \equiv (\partial^2\Psi/\partial^2 t)|_{\Sigma}, \quad (32)$$

and $\mathcal{B}_{11} : \mathcal{H}_X \rightarrow \mathcal{H}_X$, $\mathcal{B}_{12} : L^2(\Sigma, \mu) \rightarrow \mathcal{H}_X$, etc.

Since the domain of \mathcal{B} contains \mathcal{V} , eqs. (30) and (31) are guaranteed to hold for all $\psi_0 \in X$ and all $\dot{\psi}_0 \in Y$. It follows immediately that the restriction of \mathcal{B}_{11} to X vanishes and that the restriction of \mathcal{B}_{12} to $Y \subset L^2(\Sigma, \mu)$ must equal $-I$, where I denotes the identity map on functions. (Note, however, that \mathcal{B}_{12} is a map between different Hilbert spaces.) Similarly, the invariance of the dynamics under time reflection (assumption 2(ii) above) implies that the restriction of \mathcal{B}_{22} to Y must vanish. Thus, writing $B = \mathcal{B}_{21}$, we have learned that there exists an operator $B : \mathcal{H}_X \rightarrow L^2(\Sigma, \mu)$ with $X \subset \text{Dom}(B)$ such that for all $\psi_0 \in X$ we have

$$\ddot{\psi}_0 = -B\psi_0. \quad (33)$$

Comparing with eq. (2), we see that if $\psi_0 \in C_0^\infty(\Sigma)$, then $B = A$, where A was defined by eq. (3). Thus, when viewed as an operator from \mathcal{H}_X into $L^2(\Sigma, \mu)$, B is an extension of A from the domain $C_0^\infty(\Sigma)$ to a domain that includes all of X .

Finally, we note that the skew-adjointness of \mathcal{B} on \mathcal{H}_E directly implies that for all $\psi_0, \chi_0 \in X$ and all $\dot{\psi}_0, \dot{\chi}_0 \in Y$, we have

$$-(\chi_0, \dot{\psi}_0)_{\mathcal{H}_X} + (\dot{\chi}_0, B\psi_0)_{L^2} = (\psi_0, \dot{\chi}_0)_{\mathcal{H}_X} - (\dot{\psi}_0, B\chi_0)_{L^2}. \quad (34)$$

But this equation can hold if and only if for all $\psi_0 \in X$ and all $\dot{\psi}_0 \in Y$ we have

$$(\dot{\psi}_0, B\psi_0)_{L^2} = (\psi_0, \dot{\psi}_0)_{\mathcal{H}_X}. \quad (35)$$

To make further progress, we need the following lemma:

Lemma 3.2: Let $\Phi \in \mathcal{W}$ and, as before, write $\phi_t = \Phi|_{\Sigma_t}$ and $\dot{\phi}_t = (\partial\Phi/\partial t)|_{\Sigma_t}$. Then $\phi_t \in L^2(\Sigma, \mu)$ (and, hence, $X \subset L^2(\Sigma, \mu)$). Furthermore, ϕ_t is strongly differentiable with respect to t as a vector in $L^2(\Sigma, \mu)$ and $d\phi_t/dt = \dot{\phi}_t$. More generally, $d^n\phi_t/dt^n$ exists and equals $(\partial^n\Phi/\partial t^n)|_{\Sigma_t}$.

Remark: We previously showed that $Y \subset L^2(\Sigma, \mu)$ and also that $Y \subset X$. In addition, we previously showed that $T_t\Psi$ is strongly differentiable with respect to t as a vector in \mathcal{H}_E . None of the results claimed in Lemma 3.2 follow immediately from these previous results.

Proof: We label points by (t, x) as explained at the beginning of section 2. Then at each fixed x we have

$$\phi_t - \phi_0 = \int_0^t \dot{\phi}_{t'} dt' = \int_0^t [\dot{\phi}_0 + (\dot{\phi}_{t'} - \dot{\phi}_0)] dt', \quad (36)$$

and hence

$$\phi_t - \phi_0 - t\dot{\phi}_0 = \int_0^t (\dot{\phi}_{t'} - \dot{\phi}_0) dt'. \quad (37)$$

It follows that

$$\begin{aligned} |\phi_t - \phi_0 - t\dot{\phi}_0|^2 &= \left| \int_0^t (\dot{\phi}_{t'} - \dot{\phi}_0) dt' \right|^2 \\ &\leq \int_0^t |1|^2 dt' \int_0^t |\dot{\phi}_{t'} - \dot{\phi}_0|^2 dt' \\ &= t \int_0^t |\dot{\phi}_{t'} - \dot{\phi}_0|^2 dt', \end{aligned} \quad (38)$$

where the Schwartz inequality was used in the second line.

Now let $\{f_n\}$ be as in the proof of Lemma 3.1 above. We multiply eq. (38) by f_n^2 and integrate over Σ with respect to the volume element $V^{-1}d\Sigma$. We obtain

$$\begin{aligned}
\|f_n(\phi_t - \phi_0 - t\dot{\phi}_0)\|_{L^2}^2 &\leq t \int_{\Sigma} V^{-1}d\Sigma \int_0^t dt' |f_n(\dot{\phi}_{t'} - \dot{\phi}_0)|^2 \\
&= t \int_0^t dt' \int_{\Sigma} V^{-1}d\Sigma |f_n(\dot{\phi}_{t'} - \dot{\phi}_0)|^2 \\
&\leq t \int_0^t dt' \int_{\Sigma} V^{-1}d\Sigma |\dot{\phi}_{t'} - \dot{\phi}_0|^2 \\
&\leq t \int_0^t dt' E(T_{-t'}\Phi - \Phi, T_{-t'}\Phi - \Phi) \\
&\leq 4t \int_0^t dt E(\Phi, \Phi) \\
&= 4t^2 E(\Phi, \Phi).
\end{aligned} \tag{39}$$

By the same argument as used in Lemma 3.1, this inequality shows that $(\phi_t - \phi_0 - t\dot{\phi}_0) \in L^2(\Sigma, \mu)$. However, since ϕ_0 and $\dot{\phi}_0$ are in $C_0^\infty(\Sigma)$, this proves that $\phi_t \in L^2(\Sigma, \mu)$, as we desired to show.

To prove the differentiability of ϕ_t , we need to generalise and sharpen our above estimates. First, we repeat the same steps as led to eq. (39) but now use arbitrary times t' and t rather than t and 0 to find that for all t and all t' we have

$$\|\phi_{t'} - \phi_t - (t' - t)\dot{\phi}_t\|_{L^2}^2 \leq 4(t' - t)^2 E(\Phi, \Phi). \tag{40}$$

Applying this result to the solution $\partial\Phi/\partial t$, we obtain

$$\|\dot{\phi}_{t'} - \dot{\phi}_t - (t' - t)\ddot{\phi}_t\|_{L^2}^2 \leq 4(t' - t)^2 E(\partial\Phi/\partial t, \partial\Phi/\partial t), \tag{41}$$

which implies that at any fixed t , there exists a $C > 0$ and a $\delta > 0$ such that for all $|t' - t| < \delta$, we have

$$\|\dot{\phi}_{t'} - \dot{\phi}_t\|_{L^2} \leq C|t' - t|. \tag{42}$$

On the other hand, now that we know that $\phi_t \in L^2(\Sigma, \mu)$ for all t , the same steps as led to the third line of eq. (39) now yield

$$\|\phi_{t'} - \phi_t - (t' - t)\dot{\phi}_t\|_{L^2}^2 \leq (t' - t) \int_t^{t'} dt'' \|\dot{\phi}_{t''} - \dot{\phi}_t\|_{L^2}^2. \tag{43}$$

Substituting the estimate (42), we find that for all $|t' - t| < \delta$, we have

$$\left\| \phi_{t'} - \phi_t - (t' - t)\dot{\phi}_t \right\|_{L^2}^2 \leq C'(t' - t)^4. \quad (44)$$

Dividing this equation by $(t' - t)^2$ and taking the limit $t' \rightarrow t$, we immediately see that the strong derivative of ϕ_t in $L^2(\Sigma, \mu)$ exists and is equal to $\dot{\phi}_t$. The results for higher time derivatives of ϕ_t follow immediately by applying the same arguments to $\partial^n \Phi / \partial t^n$. \square

We now prove our main theorem:

Theorem 3.1: Consider any prescription for assigning a solution, Φ , to any initial data $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ such that all of the assumptions stated in section 2 are satisfied. Let $A : L^2(\Sigma, \mu) \rightarrow L^2(\Sigma, \mu)$ be the operator (3) defined on the domain $C_0^\infty(\Sigma)$. Then there exists a positive, self-adjoint extension, $A_E : L^2(\Sigma, \mu) \rightarrow L^2(\Sigma, \mu)$, of A , such that the dynamics defined by eq. (4) agrees with the dynamics given by the prescription.

Proof: We have already learned that for all $\Psi \in \mathcal{V}$, there exists an operator $B : \mathcal{H}_X \rightarrow L^2(\Sigma, \mu)$ such that eq. (33) holds. Furthermore, we know that B is an extension of A to a domain that contains X and is such that eq. (35) holds for all $\psi \in X$ and $\dot{\psi}_0 \in Y$. We also know that $Y \subset X \subset L^2(\Sigma, \mu)$.

Now let C denote the restriction of B to the domain Y and view C as a map $C : L^2(\Sigma, \mu) \rightarrow L^2(\Sigma, \mu)$ rather than a map of \mathcal{H}_X into $L^2(\Sigma, \mu)$. Since $C_0^\infty(\Sigma) \subset Y$, we see that C is densely defined on $L^2(\Sigma, \mu)$ and eq. (35) shows that, when viewed as an operator on $L^2(\Sigma, \mu)$, C is positive and symmetric on the domain Y . Therefore, there exists a positive, self-adjoint extension of C , which we shall denote as $D : L^2(\Sigma, \mu) \rightarrow L^2(\Sigma, \mu)$.

Consider, first, a solution $\Phi \in \mathcal{W}$ which is of the form $\Phi = \partial \Xi / \partial t$ for some $\Xi \in \mathcal{W}$. Then for all t we have $\phi_t = \xi_t$ so $\phi_t \in Y$ (and, of course, by definition of Y , we have $\dot{\phi}_t \in Y$ for all t). For each t , define

$$\sigma_t = \cos(D^{1/2}t)\phi_t - D^{-1/2}\sin(D^{1/2}t)\dot{\phi}_t. \quad (45)$$

Note that this formula corresponds to using the given prescription to evolve forward by t to obtain $(\phi_t, \dot{\phi}_t)$, and then using the prescription (4) with $A_E = D$ to evolve *backwards* by t . Since by Lemma 3.2 ϕ_t and $\dot{\phi}_t$ are strongly differentiable in t and since $\phi_t \in Y \subset \text{Dom}(D) \subset \text{Dom}(D^{1/2})$, it follows that σ_t is strongly differentiable in t and its derivative as a vector in $L^2(\Sigma, \mu)$ is

given by

$$\begin{aligned}\frac{d\sigma_t}{dt} = & - D^{1/2} \sin(D^{1/2}t)\phi_t + \cos(D^{1/2}t)\frac{d\phi_t}{dt} \\ & - \cos(D^{1/2}t)\dot{\phi}_t - D^{-1/2} \sin(D^{1/2}t)\frac{d\dot{\phi}_t}{dt}.\end{aligned}\quad (46)$$

However, by Lemma 3.2 we have $d\phi_t/dt = \dot{\phi}_t$. Furthermore, we have

$$\frac{d\dot{\phi}_t}{dt} = \ddot{\phi}_t = -B\phi_t = -D\phi_t, \quad (47)$$

where the first equality follows from Lemma 3.2, the second from eq. (33), and the third from the fact that D agrees with B for all vectors in Y . We therefore obtain $d\sigma_t/dt = 0$ and thus $\sigma_t = \sigma_0 = \phi_0$, i.e., we have found that

$$\phi_0 = \cos(D^{1/2}t)\phi_t - D^{-1/2} \sin(D^{1/2}t)\dot{\phi}_t. \quad (48)$$

In a similar manner, it follows that

$$\dot{\phi}_0 = \cos(D^{1/2}t)\dot{\phi}_t - D^{-1/2} \sin(D^{1/2}t)(-D\phi_t). \quad (49)$$

These equations can be inverted to yield

$$\phi_t = \cos(D^{1/2}t)\phi_0 + D^{-1/2} \sin(D^{1/2}t)\dot{\phi}_0. \quad (50)$$

This proves that all solutions of the form $\Phi = \partial\Xi/\partial t$ are as claimed in the theorem, with $A_E = D$.

Now consider a solution $\Phi \in \mathcal{W}$ arising from arbitrary initial data $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$. Let $\tilde{\Phi}$ denote the solution defined by eq. (4) with $A_E = D$. It was proven in [2] that $\tilde{\Phi}$ is a smooth solution to the Klein-Gordon equation (1) with initial data $(\phi_0, \dot{\phi}_0)$. It also follows from the analysis of [2] that $d\dot{\phi}_t/dt$ exists (in the strong Hilbert space sense in $L^2(\Sigma, \mu)$) and equals $(\partial\tilde{\Phi}/\partial t)|_{\Sigma_t}$, where $\tilde{\phi}_t \equiv \tilde{\Phi}|_{\Sigma_t}$. However, differentiation of eq. (4) yields

$$\frac{d\tilde{\phi}_t}{dt} = \cos(D^{1/2}t)\dot{\phi}_0 + D^{-1/2} \sin(D^{1/2}t)(-D\phi_0). \quad (51)$$

Eq. (51) shows that $\partial\tilde{\Phi}/\partial t$ is the solution given by eq. (4) with $A_E = D$ for the initial data $(\dot{\phi}_0, -A\phi_0)$. However, by assumption 2(i), the solution with initial

data $(\dot{\phi}_0, -A\phi_0)$ as given by our prescription is $\partial\Phi/\partial t$. Furthermore, we have just shown above that for solutions of the form $\partial\Phi/\partial t$, our prescription must agree with the prescription given by eq. (4) with $A_E = D$. It therefore follows that $\partial\tilde{\Phi}/\partial t = \partial\Phi/\partial t$. However, since $\tilde{\Phi}|_\Sigma = \Phi|_\Sigma$, it follows immediately that $\tilde{\Phi} = \Phi$, i.e., our prescription for obtaining a solution associated with arbitrary data $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ agrees with the prescription defined by eq. (4) with $A_E = D$. \square

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